e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 1

Subject:Mathematics

Title/Heading:Relationship between roots and coefficient of equation

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Relation between the roots and coefficient of equations.

Let the equation be $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$.If this equation has the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then we have

$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

$= (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$

$= x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1\alpha_2 x^{n-2} - \ldots + (-1)^n \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

$= x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^n S_n$

Where S_r is the sum of the products of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ taken r at a time.

Equating the coefficients of like powers on both sides, we have

$-p_1 = S_1$ = sum of the roots.

$(-1)^2p_2 = S_2$ = sum of the products of the roots taken two at a time.

$(-1)^3p_3 = S_3$ = sum of the products of the roots taken three at a time.

$(-1)^np_n = S_n$ = product of the roots.

If the equation is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$.

Divide each term of the equation by a_0.

The equation becomes $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$

and so we have

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

$$\alpha_1\, \alpha_2\, \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

These n equations are of no help in the general solution of an equation but they are often helpful in the solution of numerical equations when some special relation is known to exist among the roots. The method is illustrated in the examples given below.

Example 1. Show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Arithmetical progression if $2p^3 - 9pq + 27r = 0$ show that the above condition is satisfied by the equation $x^3 - 6x^2 + 13x - 10 = 0$. Hence or otherwise solve the equation.

Solution.

Let the roots of the equation $x^3 + px^2 + qx + r = 0$ be $\alpha - \delta, \alpha, \alpha + \delta$.

We have from the relation of the roots and coefficients

$$\alpha - \delta + \alpha + \alpha + \delta = -p$$

$$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$$

$$(\alpha - \delta)\alpha(\alpha + \delta) = -r.$$

Simplifying these equation, we get

$$3\alpha = -p \qquad \ldots(1)$$

$$3\alpha^2 - \delta^2 = q \qquad \ldots(2)$$

$$\alpha^3 - \alpha\delta^2 = -r. \qquad \ldots(3)$$

From (1), $\alpha = -\frac{p}{3}$.

From (2), $\delta^2 = 3\left(-\frac{p}{3}\right)^2 - q = \frac{p^2}{3} - q$.

Substituting these value in (3), we get

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

i.e., $2p^3 - 9pq + 27r = 0$.

In the equation $x^3 - 6x^2 + 13x - 10 = 0$.

$p = -6, q = 13, r = -10$.

Therefore $2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)13 + 27(-10) = 0$

The condition is satisfied and so the roots of the equation are in arithmetical progression. In this case the equations (1), (2), (3) become

$3\alpha = 6$

$3\alpha^2 - \delta^2 = 13$

$\alpha^3 - \alpha\delta^2 = 10.$

$\alpha = 2, \; 12 - \delta^2 = 13$

Therefore $\delta^2 = -1$

i.e., $\delta = \pm i$.

The roots are $2 - i, 2, 2 + i$.

Example 2. Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression.

Solution.

Let the roots of the equation be $\frac{k}{r}$, k, kr.

Therefore $\frac{k}{r} + k + kr = -\frac{3b}{a}$ .....(1)

$\frac{k^2}{r} + k^2 + k^2r = \frac{3c}{a}$ ....(2)

$k^3 = -\frac{d}{a}$ .....(3)

From (1), $k\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}$.

From (2), $k^2\left(\frac{1}{r} + 1 + r\right) = \frac{3c}{a}$.

Divided one by the other, we get $k = -\frac{c}{b}$

Substituting this value of k in (3), we get $\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$.

Therefore $ac^3 = b^3d$.

In the equation $27x^3 + 42x^2 - 28x - 8 = 0$

$\frac{k}{r} + k + kr = -\frac{42}{27}$

$\frac{k^2}{r} + k^2 + k^2r = -\frac{28}{27}$

$k^3 = \frac{8}{27}$

$\therefore\ k = \frac{2}{3}$.

Substituting the value of k in(4), we get

$\frac{2}{3}\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$

$3r^2 + 10r + 3 = 0$

$(3r + 1)(r + 3) = 0$

Therefore $r = -\frac{1}{3}$ or $r = -3$.

For both the value of r, the roots are $-2, \frac{2}{3}, -\frac{2}{9}$.

Example 3. Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression.

Solution.

Let the roots be α, β, γ.

Then $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

i.e., $2\gamma\alpha = \beta\gamma + \alpha\beta$ ……(1)

From the relation between the coefficients and the roots we have

$$\alpha + \beta + \gamma = \frac{18}{81} \quad \text{…….(2)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{36}{81} \quad \text{……..(3)}$$

$$\alpha\ \beta\gamma = -\frac{8}{81} \quad \text{….(4)}$$

From (1) and (3), we get

$$2\gamma\alpha + \gamma\alpha = -\frac{36}{81}$$

$$3\gamma\alpha = -\frac{36}{81}$$

Therefore $\gamma\alpha = -\frac{4}{27}$ …….(5)

Substituting this value of γα in (4), we get

$$\beta\left(-\frac{4}{27}\right) = -\frac{8}{81}$$

Therefore $\beta = \frac{2}{3}$.

From (2), we have

$$\alpha + \gamma = \frac{18}{81} - \frac{2}{3} = -\frac{4}{9} \quad \text{……(6)}$$

From (5)and (6), we get

$$\alpha = \frac{2}{9} \text{ and } \gamma = -\frac{2}{3}$$

The roots are $\frac{2}{9}, \frac{2}{3}$ and $-\frac{2}{3}$.

Example 4. If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Solution.

Let the roots of the equation be α, β, γ and δ

Then $\alpha + \beta = \gamma + \delta$ .....(1)

From the relation of the coefficients and the roots, we have

$\alpha + \beta + \gamma + \delta = -p$ ....(2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ ....(3)

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$ ....(4)

$\alpha\beta\gamma\delta = s$ ....(5)

From (1) and (2), we get

$2(\alpha + \beta) = -p$ ....(6)

(3) can be written as

$\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$

i.e., $(\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q$ ......(7)

(4) can be written as

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$

$(\alpha\beta + \gamma\delta)(\alpha + \beta) = -r$ ......(8)

From (6) and (7), we get

$\alpha\beta + \gamma\delta + \frac{p^2}{4} = q$

$\therefore \quad \alpha\beta + \gamma\delta = q - \frac{p^2}{4}$ .....(9)

From (8), we get

$$-\frac{p}{2}(\alpha\beta + \gamma\delta) = -r$$

$$\alpha\beta + \gamma\delta = \frac{2r}{p} \qquad \ldots\ldots\ (10)$$

Equating (9) and (10), we get

$$q - \frac{p^2}{4} = \frac{2r}{p}$$

$$4pq - p^3 = 8r$$

$$p^3 + 8r = 4pq.$$

Example 5. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.

Solution.

Let the roots of the equation be α, β, γ and δ

Here $\gamma = -\delta$

i.e., $\gamma + \delta = 0$ ……(1)

From the relation of the roots and coefficients

$$\alpha + \beta + \gamma + \delta = 2 \qquad \ldots\ldots(2)$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 4 \qquad \ldots\ldots(3)$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -6 \qquad \ldots\ldots(4)$$

$$\alpha\beta\gamma\delta = -21 \qquad \ldots\ldots(5)$$

from (1) and (2), we get $\alpha + \beta = 2$ ……(6)

(3) can be written as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 4$

$\alpha\beta + \gamma\delta = 4$ ……(7)

(4) can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -6$

$\gamma\delta(\alpha + \beta) = -6$ ....(8)

from (6) and (8), we get $\gamma\delta = -3$ .....(9)

but $\gamma + \delta = 0$ $\therefore$ $\gamma = \sqrt{3}$, $\delta = -\sqrt{3}$.

From (7) and (9), we get $\alpha\beta = 7$

$\therefore$ α and β are the roots of x2 – 2x + 7 = 0.

$\therefore \alpha = 1 + \sqrt{-6}$, $\beta = 1 - \sqrt{-6}$

Therefore the roots of the equation are $\pm\sqrt{3}$, $1 \pm\sqrt{-6}$.

Example 6. Find the condition that the general bi quadratic equation $ax^4 + 4bx^3 + 6cx^2 +$ $+ e = 0$ may have two pairs of equal roots.

Solution.

Let the roots be α, α, β, β.

From the relations of coefficients and roots

$2\alpha + 2\beta = -\frac{4b}{a}$ ....(1)

$\alpha^2 + \beta^2 + 4\alpha\beta = \frac{6c}{a}$ ....(2)

$2\alpha\beta^2 + 2\alpha^2\beta = -\frac{4d}{a}$ ....(3)

$\alpha^2\beta^2 = \frac{e}{a}$ ....(4)

From (1), we get $\alpha + \beta = -\frac{2b}{a}$ ....(5)

From (3), we get $2\alpha\beta(\alpha + \beta) = -\frac{4d}{a}$

$\therefore \alpha\beta = \frac{d}{b}$ ....(6)

From (5) and (6), we get that α, β are the roots of the equation $x^2 + \frac{2b}{a}x + \frac{d}{b} = 0$

$\therefore ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2$

Comparing coefficients

$$6c = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}$$

$$\therefore 3abc = a^2d + 2b^3 \text{ and } eb^2 = ad^2.$$

Exercises